Sparse matrix optimization problems in computational color imaging

iPAL Group Meeting

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What is a sparse matrix?

- A matrix with significantly more number of 0’s than non-zero elements

- In the words of J. H. Wilkinson, “… any matrix with enough zeros that it pays to take advantage of them”

- Structure lends itself to special techniques which improve computational efficiency: time and storage

- Examples: diagonal matrix (special case: identity matrix), band matrix

- Scenarios where sparse matrices appear: computational fluid dynamics (solution to partial differential equations), robotics, financial modeling, networks, computational color imaging
Examples from computational color imaging

- Multidimensional look-up-table design on simplex topologies\(^1\)
- Image spatial resolution enhancement via multiple image captures\(^2\)

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\(^1\)Monga and Bala, ICASSP 2010
\(^2\)Monga and Srinivas, Invited paper, ASILOMAR Conference 2010
Example 1: Color look-up-table design

- The goal of color transformation: keep color properties during transformation

Figure: Color changed between different devices
Look Up Table

- In many cases, color transformations can be seen as a mapping. We can use LUT.
- Precompute the transform for all inputs and store corresponding outputs in look up table (LUT)

**Full-Resolution LUT**
- Perfectly accurate transformation
- But, impractical because constraints of storage
- CMYK → CIELAB need 16GB of storage

**Sparse LUTs** are employed, multidimensional interpolation for input is used.
- Trade-off between cost and accuracy
- Choose the node locations and output values
Node locations and Output values

Figure: Optimization of Node locations and Output values

\[3\text{Monga et al., ICASSP 2010}\]
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Objective function

Choose look-up-table nodes $\chi^{nd}$ and corresponding output values $y_{j}^{nd}$:

$$C(\chi^{nd}, y^{nd}) = \sum_{i=1}^{t} (y_{i} - \hat{f}(x_{i}, \chi^{nd}, y^{nd}))^2$$

Where Weight matrix $W_{\chi^{nd}}$ induced by node locations $\chi^{nd}$

$$\hat{f}(x, \chi^{nd}, y^{nd}) = \sum_{x_j^{nd} \in N(x_i)} w_j y_{j}^{nd}$$

$$(\chi_{opt}^{nd}, y_{opt}^{nd}) = \arg \min_{(\chi^{nd}, y^{nd})} ||y^{tr} - W_{\chi^{nd}}y^{nd}||^2$$

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Optimization problem

\[
\begin{align*}
\text{minimize} & \quad \| y^{tr} - W^{\chi_{nd}} y^{nd^*} \|^2 \\
\text{subject to} & \quad w_i^T e_i = 0, i = 1, 2, \ldots, K \\
& \quad w_{i,j} \geq 0, i = 1, 2, \ldots, T, j = 1, 2, \ldots, M \\
& \quad W^{\chi_{nd}} \cdot 1 = 1.
\end{align*}
\]

(1)

Interpretation of constraints:

- \( w_i \) is the \( i \)-th row of \( W^{\chi_{nd}} \), \( e_i \) is a membership vector of zeros and ones. The membership constraint fixes the locations of the nodes with non-zero weights.

- \( w_{i,j} \geq 0 \): non-negativity of individual matrix elements, since interpolation weights are non-negative.

- Sum of elements in each row of \( W^{\chi_{nd}} \) is 1, since it is an interpolation matrix.

\( W^{\chi_{nd}} \) is a \textbf{sparse matrix}?.
Example 2: Spatial resolution enhancement

Figure: Digital image acquisition system\(^4\).

\(^4\)Park et al., IEEE Signal Processing Magazine, 2003
Model of the forward imaging process

\[ y_k = DBT_k x + n_k, \quad 1 \leq k \leq K \]  (2)

where

- \( x \in \mathbb{R}^n \) is the unknown high-resolution image
- \( y_k \in \mathbb{R}^m \) \((m < n)\) represents the \( k \)-th low-resolution image
- \( T_k \in \mathbb{R}^{n \times n} \) is the \( k \)-th geometric warping matrix
- \( B \in \mathbb{R}^{n \times n} \) describes camera optical blur
- \( D \in \mathbb{R}^{m \times n} \) is a downsampling matrix of 1s and 0s
- \( n_k \in \mathbb{R}^m \) is the noise vector that corrupts \( y_k \).

**Goal:** Reconstruct an estimate of \( x \) given \( y_k \), under limited knowledge of imaging model parameters.
Optimization problem

minimize \[ \sum_{k=1}^{K} \| y_k - DBT_k x \|_p + \lambda \rho(x) \]

subject to \[ 0 \leq x \leq 1 \]
\[ 0 \leq DBT_k x \leq 1, \quad 1 \leq k \leq K \]
\[ T_k \cdot 1 = 1, \quad 1 \leq k \leq K \]
\[ B \cdot 1 = 1 \]
\[ t_{k,i}^T m_{k,i} = 0, \quad 1 \leq i \leq n, \quad 1 \leq k \leq K \]
\[ b_i^T e_i = 0, \quad 1 \leq i \leq n \]

D, B and T_k are sparse matrices?.
Sparsity of \( W \) matrix

Sparsity is induced by membership constraints

- Each row of \( W \) indicates location of non-zero values.
- Corresponding column in membership matrix \( E \): zeros at locations matching the \( W \), ones elsewhere.

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Common framework

minimize $\|y - Wx\|^2$

subject to 

\begin{align*}
  w_i^T e_i &= 0, \quad i = 1, 2, \ldots, K \\
  w_{i,j} &\geq 0, \quad i = 1, 2, \ldots, T, j = 1, 2, \ldots, M \\
  W.1 &= 1.
\end{align*}

(4)

constraints:

- (1) Membership Constraint: $w_i^T e_i = 0, \quad i = 1, 2, \ldots, K$ fixes particular elements of $W$ must be zero.

- (2) Non-negative: $w_{i,j} \geq 0$: non-negativity of $W$ matrix elements.

- (3) Interpolation Constraint: $W.1 = 1$, satisfy interpolation requirement.

$W$ is a **sparse matrix**.
Separable convexity of cost function

Convexity in $W$ for fixed $x$ (using triangle inequality):

$$
\|y(\alpha W_1 + (1 - \alpha)W_2)x\|_p
= \|\alpha y + (1 - \alpha)y - \alpha W_1x - (1 - \alpha)W_2x\|_p
= \|\alpha(y - W_1x) + (1 - \alpha)(y - W_2x)\|_p
\leq \alpha\|y - W_1x\|_p + (1 - \alpha)\|y - W_2x\|_p
$$
Formulation as a QP

\[
\begin{align*}
\arg\min_W \|y - Wx\|^2 &= \arg\min_W ((y - Wx)^T(y - Wx)) \\
&= \arg\min_W (x^T W^T Wx - 2y^T Wx)
\end{align*}
\]

Let \( Y := xx^T \) and \( w_i^T \) denote the \( i \)-th row of \( W \). Then,

\[
x^T W^T Wx = tr(x^T W^T Wx) = tr(Wxx^T W^T) \\
= tr(WYW^T) = \sum_{i=1}^{K} w_i^T Yw_i \\
= vec(W^T)^T(I_m \otimes Y)vec(W^T) \\
= vec(W^T)^T \tilde{Y} vec(W^T),
\]

where \( vec(\cdot) \) is the vectorizing operator, \( I_m \) is the \( m \times m \) identity matrix, and \( \otimes \) represents the Kronecker product.
Formulation as a QP (contd.)

Similarly,

\[ y^T W x = tr(y^T W x) = tr(W x y^T) \]
\[ = tr((x y^T)^T W^T) = vec(x y^T)^T vec(W^T). \]

With \( z := vec(W^T) \) and \( c := -2vec(x y^T) \), the original cost function becomes

\[ \arg \min_z z^T \tilde{Y} z + c^T z. \tag{5} \]

- \( Y \) positive semidefinite, \( I_m \) positive definite \( \Rightarrow \tilde{Y} \) is positive semidefinite (Kronecker product preserves positive definiteness).

- Cost function is quadratic in \( z \).
What about the constraints?

- **Membership constraint:** Let \( \mathbf{0} \in \mathbb{R}^M \) denote the vector with all zeros. Define \( \mathbf{E} \in \mathbb{R}^{K \times KM} \) such that

\[
\mathbf{E} = \begin{bmatrix}
\mathbf{e}_1^T & \mathbf{0}^T & \mathbf{0}^T & \cdots & \mathbf{0}^T \\
\mathbf{0}^T & \mathbf{e}_2^T & \mathbf{0}^T & \cdots & \mathbf{0}^T \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0}^T & \mathbf{0}^T & \cdots & \mathbf{0}^T & \mathbf{e}_K^T
\end{bmatrix}
\]  

(6)

\[
\mathbf{w}_i^T \mathbf{e}_i = 0, \ i = 1, 2, \ldots, K \iff \mathbf{Ez} = \mathbf{0}(\in \mathbb{R}^K).
\]  

(7)

- **Non-negativity:**

\[
\mathbf{w}_{i,j} \geq 0 \iff \mathbf{z} \succeq \mathbf{0}(\in \mathbb{R}^{KM}).
\]  

(8)

- **Interpolation constraint:** Let \( \mathbf{1} \in \mathbb{R}^M \) denote the vector with all ones. Define \( \mathbf{F} \in \mathbb{R}^{K \times KM} \) such that

\[
\mathbf{F} = \begin{bmatrix}
\mathbf{1}^T & \mathbf{0}^T & \mathbf{0}^T & \cdots & \mathbf{0}^T \\
\mathbf{0}^T & \mathbf{1}^T & \mathbf{0}^T & \cdots & \mathbf{0}^T \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0}^T & \mathbf{0}^T & \cdots & \mathbf{0}^T & \mathbf{1}^T
\end{bmatrix}
\]  

(9)

\[
\mathbf{W} \cdot \mathbf{1} = \mathbf{1} \iff \mathbf{Fz} = \mathbf{1}(\in \mathbb{R}^K).
\]  

(10)
Complete optimization problem

minimize $z^T \tilde{Y} z + c^T z$

subject to

\[ \begin{align*}
Ez &= 0 \\
z &\succeq 0 \\
Fz &= 1
\end{align*} \tag{11} \]

where

- $z = vec(W^T)$
- $\tilde{Y} = I_m \otimes (xx^T)$
- $c = -2vec(xx^T)$
- $E \in \mathbb{R}^{K \times KM}$ is defined in Eq. (6)
- $F \in \mathbb{R}^{K \times KM}$ is defined in Eq. (9)
- $K$ is the number of training points.
Result: color look up table
Numerical result

<table>
<thead>
<tr>
<th>Node Location</th>
<th>Node value</th>
<th># of Nodes</th>
<th>Avg. $\Delta E$</th>
<th>95% $\Delta E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>Hi-res LUT</td>
<td>4096</td>
<td>6.2</td>
<td>8.2</td>
</tr>
<tr>
<td>SSD as in [5]</td>
<td>Hi-res LUT</td>
<td>4096</td>
<td>4.1</td>
<td>7.4</td>
</tr>
<tr>
<td>Uniform</td>
<td>Least-squares</td>
<td>4096</td>
<td>5.6</td>
<td>7.89</td>
</tr>
<tr>
<td>SOLVE-L</td>
<td>SOLVE-L</td>
<td>4096</td>
<td>3.11</td>
<td>6.1</td>
</tr>
<tr>
<td>SOLVE-G</td>
<td>SOLVE-G</td>
<td>1800</td>
<td>2.95</td>
<td>6.1</td>
</tr>
<tr>
<td>RESOLVE</td>
<td>RESOLVE</td>
<td>1800</td>
<td>1.89</td>
<td>4.83</td>
</tr>
</tbody>
</table>
Results: Image resolution enhancement

Figure: Sample low-resolution images.

Figure: (a) Bilinear interpolation, (b) Heuristic estimate of $W$ (no optimization), (c) Practical upper bound: reconstructed from true imaging model parameters, (d) $W$ optimization.
Future work

- Dual function/ Lagrangian analysis
- Does a closed-form solution to the optimization problems exist?
  - Can dropping or introducing some of the constraints help?
  - Will the resulting solution be physically meaningful?
- Efficient numerical optimization